[0.5] 1. Prove that $(-1)^1 + (-1)^2 + (-1)^3 + \cdots + (-1)^n = \dfrac{(-1)^n - 1}{2}$, for any $n \in \mathbb{N}$.

- Base step: $(-1)^1 = -1 = \dfrac{(-1)^1 - 1}{2}$ ✓
- Induction step: assume that $\displaystyle\sum_{j=1}^{k}(-1)^j = \frac{(-1)^k - 1}{2}$ for some $k \in \mathbb{N}$. Then:

$$\begin{aligned}
\sum_{j=1}^{k}(-1)^j + (-1)^{k+1} &= \frac{(-1)^k - 1}{2} + (-1)^{k+1} \\
&= \frac{(-1)^k - 1 + 2(-1)^{k+1}}{2} \\
&= \frac{(-1)^k - 1 + (-1)^{k+1} + (-1)^{k+1}}{2} \\
&= \frac{(-1)^k - 1 - (-1)^k + (-1)^{k+1}}{2} \\
&= \frac{(-1)^{k+1} - 1}{2}.
\end{aligned}$$

- Conclusion: it follows from the principle of mathematical induction that $(-1)^1 + (-1)^2 + (-1)^3 + \cdots + (-1)^n = \dfrac{(-1)^n - 1}{2}$, for any $n \in \mathbb{N}$.

[0.5] 2. Find all complex numbers $z \in \mathbb{C}$ that satisfy the equation $\left|z + \dfrac{1+\imath}{1-\imath}\right| = \left|z - \dfrac{1+\imath}{1-\imath}\right|$ and plot them in the Argand plane.

First we see that:

$$\frac{1+\imath}{1-\imath} = \frac{1+\imath}{1-\imath} \cdot \frac{1+\imath}{1+\imath} = \imath.$$

Therefore:

$$\begin{aligned}
\left|z + \frac{1+\imath}{1-\imath}\right| = \left|z - \frac{1+\imath}{1-\imath}\right| \iff |z + \imath| &= |z - \imath| \\
|a + (b+1)\imath| &= |a + (b-1)\imath| \\
a^2 + (b+1)^2 &= a^2 + (b-1)^2 \\
a^2 + b^2 + 2b + 1 &= a^2 + b^2 - 2b + 1 \\
4b &= 0 \\
b &= 0.
\end{aligned}$$

Therefore, the solution is $z = a$, with $a \in \mathbb{R}$, or all real numbers. The plot is simply the real line.

[0.5] 3. Prove rigorously, using the $(\varepsilon, \delta)$-definition of limit, that $\lim_{x\to 0} \frac{x^2-2x+1}{x-1} = -1$.

– Preliminary analysis: First we notice that, due to continuity and together with $\lim_{x\to c} \frac{f(x)}{f(x)} = 1$ for arbitrary functions, $\lim_{x\to 0} \frac{x^2-2x+1}{x-1} = -1 \iff \lim_{x\to 0} \frac{(x-1)^2}{x-1} = -1 \iff \lim_{x\to 0}(x-1) = -1$. This means that it suffices to prove $\lim_{x\to 0}(x-1) = -1$. Now it is straightforward to choose $\delta = \varepsilon$.

*Proof.* For all $\varepsilon > 0$, let $\delta > 0$. If $0 < |x| < \delta$, then $|x-1+1| = |x-1-(-1)| < \delta = \varepsilon$.

Therefore, from the $(\varepsilon, \delta)$-definition of limit it follows that $\lim_{x\to 0}(x-1) = -1$ and therefore

$$\lim_{x\to 0} \frac{x^2-2x+1}{x-1} = -1.$$

□

[0.75] 4. Evaluate the limit $\lim_{x\to 0} \frac{\sin x - 2\cos x - x + 2}{x^2}$ **without using L'Hospital's rule**.

We know from the lectures that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Therefore:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x - 2\cos x - x + 2}{x^2} &= \lim_{x\to 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots) - 2(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots) - x + 2}{x^2} \\ &= \lim_{x\to 0} \frac{2\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}{x^2} \\ &= 1.\end{aligned}$$

[0.75] 5. Let $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Prove that for every $k \geq 1$, the equation

$$x^{2k+1} + a_{2k}x^{2k} + a_{2k-1}x^{2k-1} + \cdots + a_2x^2 + a_1x + a_0 = 0,$$

where all the coefficients $a_i$, $i = 0, 1, \ldots, 2k$, are real numbers, has at least one real solution.

Let $p(x) = x^{2k+1} + a_{2k}x^{2k} + a_{2k-1}x^{2k-1} + \cdots + a_2x^2 + a_1x + a_0$. Since $p(x)$ is an odd polynomial, there exist $\alpha < 0$ and $\beta > 0$ such that $p(\alpha) < 0$ and $p(\beta) > 0$.

It follows from the intermediate value theorem that there exists a $c \in (\alpha, \beta)$ such that $p(c) = 0$.

[0.75] 6. Suppose that $f(x)$ is continuous on a closed interval $[a,b]$ and that $f(x)$ is differentiable on the open interval $(a,b)$, with $a<b$. Prove that if $f'(x)<0$ for all $x\in(a,b)$, then the function $f(x)$ is strictly decreasing on the interval $[a,b]$.

We shall proceed by contradiction: assume there exist $x_1,x_2\in(a,b)$, with $x_1<x_2$ and such that $f(x_2)-f(x_1)\geq 0$. On the other hand, we know from the mean value theorem that there exists a $c\in(x_1,x_2)$ such that $f'(c)=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}\geq 0$. But this is a contradiction of $f'(x)<0$ for all $x\in(a,b)$. Therefore, we conclude that for any $x_1,x_2\in(a,b)$, with $x_1<x_2$ it holds that $f(x_2)-f(x_1)<0$ which is equivalent to $f(x_1)>f(x_2)$.

[0.5] 7. Let $f(x)$ and $g(x)$ be differentiable functions. Find the derivative of the function $h(x)=f(x)^{g(x)}$.

From now on we assume that the images of the functions are such that the following operations are well-defined.

$$h(x)=f(x)^{g(x)} \iff \ln h(x)=g(x)\ln f(x)$$

$$\rightarrow \frac{h'(x)}{h(x)}=g'(x)\ln f(x)+\frac{g(x)f'(x)}{f(x)}$$

$$\rightarrow \boxed{h'(x)=h(x)\left(g'(x)\ln f(x)+\frac{g(x)f'(x)}{f(x)}\right)}$$

[0.5] 8. Evaluate the limit $\lim\limits_{x\to\infty}(e^x+2x)^{1/x}$.

Let $y=(e^x+2x)^{1/x}$, then $\ln y=\dfrac{1}{x}\ln(e^x+2x)$.

So:

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{1}{x}\ln(e^x+2x)\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{e^x+2}{e^x+2x}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{e^x}{e^x+2}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{e^x}{e^x}=1.$$

This implies that $\lim\limits_{x\to\infty}(e^x+2x)^{1/x}=e$.

[0.5] 9. Let $f(x)$, $g(x)$ and $h(x)$ be differentiable functions. Evaluate the integral

$$\int_a^b (f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x))\mathrm{d}x.$$

We notice that $f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)=(f(x)g(x)h(x))'$. Therefore

$$\int_a^b (f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x))\mathrm{d}x=\int_a^b (f(x)g(x)h(x))'\mathrm{d}x=f(b)g(b)h(b)-f(a)g(a)h(a),$$

where we have used the fundamental theorem of calculus.

[0.75] 10. Evaluate the following integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 \cos x \sin^2 x + x \sin x\right) \mathrm{d}x$.

Since $x^3 \cos x \sin^2 x$ is an odd function, it immediately follows that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 \cos x \sin^2 x + x \sin x\right) \mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x \sin x)\, \mathrm{d}x.$$

Integrating by parts we have that $\int x \sin x \mathrm{d}x = -x \cos x + \sin x$. Therefore:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 \cos x \sin^2 x + x \sin x\right) \mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x \sin x)\, \mathrm{d}x = (-x \cos x + \sin x)\Big|_{-\pi/2}^{\pi/2} = 2.$$

[0.75] 11. Find the average of the function $f(t) = \frac{1}{\sqrt{2-t}}$ on the interval $[0, 2]$.

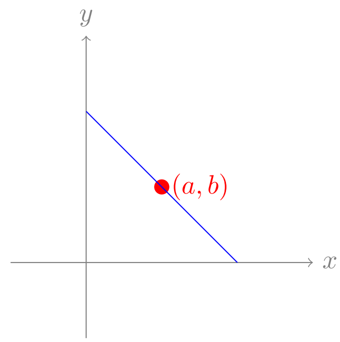
By definition:

$$f_{avg} = \frac{1}{2} \int_0^2 \frac{1}{\sqrt{2-t}} \mathrm{d}t,$$

We notice that $\int_0^2 \frac{1}{\sqrt{2-t}} \mathrm{d}t$ is improper. Therefore:

$$\begin{aligned} f_{avg} &= \frac{1}{2} \lim_{x \to 2^-} \int_0^x \frac{1}{\sqrt{2-t}} \mathrm{d}t \\ &= \frac{1}{2} \lim_{x \to 2^-} \left(-2\sqrt{2-t}\right)\Big|_0^x \\ &= \lim_{x \to 2^-} \left(-\sqrt{2-x} + \sqrt{2}\right) \\ &= \sqrt{2}. \end{aligned}$$

[0.75] 12. Let $a$ and $b$ be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point $(a, b)$.




The line that passes through $(a, b)$ is given by $y = m(x - a) + b$, where $m$ is the slope of the line. From this we know that the length of the line segment is a function of $m$.

Of course there is an infinite number of lines that pass through $(a, b)$ and that are cut-off by the first quadrant. However, from the sketch we observe that if the line is going to have finite length, then $m < 0$.

Notice that if $y = 0$, then $x = \dfrac{ma - b}{m}$, while when $x = 0$, then $y = b - ma$. This means that the length of the line that is cut off by the first quadrant and passes through the point $(a, b)$ is:

$$l(m) = \sqrt{\left(\frac{ma - b}{m}\right)^2 + (b - ma)^2}.$$

Minimizing $l(m)$ is equivalent to minimizing $L(m) = l(m)^2 = \left(\dfrac{ma - b}{m}\right)^2 + (b - ma)^2 = (ma - b)^2\left(1 + \dfrac{1}{m^2}\right)$, so now we only focus on minimizing $L(m)$. So:

$$\begin{aligned} L'(m) &= 2a(ma - b)\left(1 + \frac{1}{m^2}\right) - \frac{2}{m^3}(ma - b)^2 = 2(ma - b)\left(a + \frac{a}{m^2} - \frac{a}{m^2} + \frac{b}{m^3}\right) \\ &= 2(ma - b)\left(a + \frac{b}{m^3}\right) = \frac{2(ma - b)}{m^3}(am^3 + b). \end{aligned}$$

Notice at this point that since $\dfrac{ma - b}{m^3} > 0$. That implies that since $m < 0$, the only solution for $L'(m) = 0$ is $m = -\left(\dfrac{b}{a}\right)^{1/3}$. Moreover, the sign of $L'(m)$ is exclusively given by the sign of $(am^3 + b)$.

Next, we notice that $L'(m) < 0$ for $m < -\left(\dfrac{b}{a}\right)^{1/3}$ and $L'(m) > 0$ for $m > -\left(\dfrac{b}{a}\right)^{1/3}$, which means that the value of $m$ that we have found corresponds to a minimum of $L(m)$.

Finally, let us define $m^* = -\left(\dfrac{b}{a}\right)^{1/3}$. Thus, the length of the shortest line segment that is cut off by the first quadrant and passes through the point $(a, b)$ is $l(m^*)$, that is:

$$\begin{aligned} l(m^*) &= \sqrt{\left(\frac{m^*a - b}{m^*}\right)^2 + (b - m^*a)^2} = \sqrt{\frac{(m^*a - b)^2 + (m^*)^2(b - m^*a)^2}{(m^*)^2}} = \sqrt{\frac{(m^*a - b)^2(1 + (m^*)^2)}{(m^*)^2}} \\ &= \frac{m^*a - b}{m^*}\sqrt{1 + (m^*)^2} = \frac{b^{1/3}a^{2/3} + b}{b^{1/3}a^{-1/3}}\sqrt{1 + \frac{b^{2/3}}{a^{2/3}}} = \frac{b^{1/3}a^{2/3} + b}{b^{1/3}}\sqrt{a^{2/3} + b^{2/3}} \\ &= (a^{2/3} + b^{2/3})\sqrt{a^{2/3} + b^{2/3}} = \boxed{(a^{2/3} + b^{2/3})^{3/2}} \end{aligned}$$

[0.5] 13. Find the length of the curve $y = 1 + e^{-x}$ for $0 \le x \le 1$.

$y' = -e^{-x} \to 1 + (y')^2 = 1 + e^{-2x}$, so:

$$L = \int_0^1 \sqrt{1 + e^{-2x}}\mathrm{d}x$$

Using $u = e^{-x}$ we have:

$$L = -\int_1^{e^{-1}} \frac{\sqrt{1+u^2}}{u}\mathrm{d}u$$

We can now use the formula $\int \frac{\sqrt{a^2+x^2}}{a}\mathrm{d}x = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$, which appears in the back of the book (you of course could have solved this integral by trigonometric substitution as well), to obtain:

$$\begin{aligned} L &= \left(\ln\frac{1+\sqrt{1+u^2}}{u} - \sqrt{1+u^2}\right)\Big|_1^{e^{-1}} \\ &= \ln\frac{1+\sqrt{1+e^{-2}}}{e^{-1}} - \sqrt{1+e^{-2}} - \ln(1+\sqrt{2}) + \sqrt{2} \\ &= \ln(1+\sqrt{1+e^{-2}}) - \ln(e^{-1}) - \sqrt{1+e^{-2}} - \ln(1+\sqrt{2}) + \sqrt{2} \\ &= \ln(1+\sqrt{1+e^{-2}}) + 1 - \sqrt{1+e^{-2}} - \ln(1+\sqrt{2}) + \sqrt{2}. \end{aligned}$$

[0.5] 14. Solve the initial value problem $y' = \frac{2x(y^3+1)}{y^2}$, $y(1) = 1$.

$$\begin{aligned} \frac{y^2}{y^3+1}\mathrm{d}y &= 2x\mathrm{d}x \\ \text{substitution: } u &= y^3+1 \\ \frac{1}{3}\int\frac{\mathrm{d}u}{u} &= 2\int x\mathrm{d}x \\ \frac{1}{3}\ln|u| &= x^2 + C \\ \ln|u| &= 3x^2 + D \\ u &= Ae^{3x^2} \\ y^3 &= Ae^{3x^2} - 1 \\ y &= \left(Ae^{3x^2} - 1\right)^{1/3}, \end{aligned}$$

where $A \neq 0$. However, we notice that if $A = 0$, then $y = -1$, which satisfies the differential equation. So, we have that the general solution to the given differential equation is $\boxed{y(x) = \left(Ae^{3x^2} - 1\right)^{1/3} \text{ with } A \in \mathbb{R}}$.

We now look for the solution of the initial value problem: $y(1) = (Ae^3 - 1)^{1/3} = 1$, which implies $A = 2e^{-3}$. Therefore, the solution of the given initial value problem is $\boxed{y(x) = \left(2e^{-3}e^{3x^2} - 1\right)^{1/3}}$.

[0.5] 15. Solve the differential equation $xy' + 2y = x^2 - x + 2$, and state for which values of $x$ is the solution well-defined. (That is, state the domain of the solution)

First we write the given equation as a linear first order differential equation:

$$y' + \frac{2}{x}y = x - 1 + \frac{2}{x},$$

and compute that the integrating factor is

$$I(x) = x^2.$$

Then

$$\begin{aligned}(x^2y)' &= x^3 - x^2 + 2x \\ x^2y &= \frac{x^4}{4} - \frac{x^3}{3} + x^2 + C \\ y &= \frac{x^2}{4} - \frac{x}{3} + 1 + \frac{C}{x^2}.\end{aligned}$$

For arbitrary constant $C$, we notice that the solution is well defined for all $x \neq 0$. Only in the particular case where $C = 0$ is the solution defined for all $x \in \mathbb{R}$.

[1 bonus] 16. Consider a fixed population of $N > 0$ individuals. In epidemiology, the so-called $SIS$ model is given by the equations:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= -\frac{\beta}{N}SI + \gamma I \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta}{N}SI - \gamma I, \end{aligned} \tag{1}$$

where $S = S(t)$ denotes the number of Susceptible individuals at time $t$, and $I = I(t)$ denotes the number of Infected individuals at time $t$. Because the total population $N$ is fixed, the conservation law $S(t) + I(t) = N$ holds for all $t$. Moreover, for the model one considers $\beta > 0$ and $\gamma > 0$.

a) Using the conservation law and (1), obtain a single differential equation that models the evolution of the Infected individuals over time. [**Hint:** this equation should be of the form $\frac{\mathrm{d}I}{\mathrm{d}t} = f(I, N, \beta, \gamma)$]

b) Solve the equation obtained in item a). That is, find the function $I(t)$ that satisfies the differential equation of item a).

c) Prove, using the solution obtained in b) that: if $\frac{\beta}{\gamma} < 1$, then $\lim_{t\to\infty} I(t) = 0$. Give an interpretation of this result.

d) Assuming $\frac{\beta}{\gamma} > 1$, what is $\lim_{t\to\infty} I(t)$? Give an interpretation of this result.

a) We substitute $S = N - I$ in (1) and obtain:

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta}{N}(N - I)I - \gamma I \\ &= (\beta - \gamma)I - \frac{\beta}{N}I^2 \end{aligned}$$

b) To simplify notation, let $a = \beta - \gamma$ and $b = \frac{\beta}{N}$. Then we have:

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= aI - bI^2 \\ &= (a - bI)I \\ \frac{\mathrm{d}I}{I(a - bI)} &= \mathrm{d}t. \end{aligned}$$

Using partial fractions we find that

$$\frac{1}{I(a - bI)} = \frac{1}{aI} + \frac{b}{a(a - bI)}.$$

Integrating:

$$\begin{aligned}
\int \frac{\mathrm{d}I}{I(a-bI)} &= \int \mathrm{d}t \\
\frac{1}{a}\int \frac{\mathrm{d}I}{I} + \frac{b}{a}\int \frac{\mathrm{d}I}{a-bI} &= t + C \\
\frac{1}{a}\ln|I| - \frac{1}{a}\ln|a-bI| &= t + C \\
\ln\left|\frac{I}{a-bI}\right| &= at + D \\
\frac{I}{a-bI} &= Ae^{at},
\end{aligned}$$

where $A \in \mathbb{R}$. This solution can be written in explicit form as:

$$I(t) = \frac{Aae^{at}}{1 + Abe^{at}}.$$

Notice that (this will be used later):

- $*$ if $a < 0$, then $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{Aae^{at}}{1+Abe^{at}} = 0$
- $*$ if $a > 0$, then $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{Aae^{at}}{1+Abe^{at}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{Aa^2e^{at}}{Aabe^{at}} = \frac{a}{b}$

c) If $\frac{\beta}{\gamma} < 1$ then $a = \beta - \gamma < 0$ and the proof follows from the first of the above limits. The interpretation is that if $\frac{\beta}{\gamma} < 1$ then the infection disappears.

d) If $\frac{\beta}{\gamma} > 1$ then $a = \beta - \gamma > 0$ and from the second of the above limits we have that $\lim_{t\to\infty} I(t) = \frac{a}{b} = \frac{N(\beta-\gamma)}{\beta}$. The interpretation is that if $\frac{\beta}{\gamma} > 1$, then the number of infections tends, in the long run, to a fixed value of $\frac{N(\beta-\gamma)}{\beta}$.